

Graviton Amplitudes from Collinear Limits of Gauge Amplitudes

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Abstract

We express all tree-level graviton amplitudes in Einstein’s gravity as the collinear limits of a linear combination of pure Yang-Mills amplitudes in which each graviton is represented by two gauge bosons, each of them carrying exactly one half of graviton’s momentum and helicity.

The amplitudes describing scattering processes of many gravitons in quantized Einstein’s general relativity are related to the amplitudes describing vector gauge boson scattering in Yang-Mills theory. As discovered by Kawai, Lewellen and Tye (KLT) [1] in 1985, gravitational amplitudes can be written as “squares” of gauge amplitudes, more precisely as bilinear forms of partial gauge amplitudes weighted by the kinematic coefficients presently known as the “KLT kernel.” Kawai, Lewellen and Tye discovered these relations in the framework of string theory, as a connection between closed and open string amplitudes. Their quadratic form is a direct consequence of the factorization of the graviton vertex operator into the operators creating non-interacting left- and right-moving fluctuations of the world-sheet. They support an intuitive picture of the closed string as a loop of two open strings connected at the ends [2]. More recently, Cachazo, He and Yuan [3,4,5], developed a novel representation of graviton amplitudes, by utilizing the so-called scattering equations. Here again, gravity appears as a “square” of gauge theory.

In a recent work [6], we expressed tree-level Einstein-Yang-Mills amplitudes involving one graviton and an arbitrary number of gauge bosons as *linear* combinations of pure Yang-Mills amplitudes in which the graviton appears as a pair of collinear vector bosons, each of them carrying exactly one-half of its momentum and helicity. This result is a low energy, field theory manifestation of a much broader class of linear relations between closed and open string amplitudes, which will be discussed elsewhere [7]. It indicates that, in some way, the graviton can be considered as a pair of gauge bosons beyond the world-sheet, as a bound state in physical space-time. Although Weinberg-Witten theorem [8] rules out a massless spin 2 graviton emergent from pure gauge dynamics¹, it is possible that such linear relations reflect something more subtle.

In the present work, we express the N -graviton amplitude as a collinear limit of a particular linear combination of pure Yang-Mills (partial) $2N$ -gluon² amplitudes. The paper is organized in a simple way. First, we establish notation. Then we will state the result and prove it by showing agreement with the KLT formula. As an illustration, we will work out explicitly the three-graviton case.

The momenta k_i and helicities $\lambda_i = \pm 2$ of $N-1$ gravitons will be split into momenta p_i, q_i and helicities μ_i, ν_i of $2(N-1)$ gluons in the following way:

$$p_i = q_i = \frac{k_i}{2}, \quad \mu_i = \nu_i = \frac{\lambda_i}{2}, \quad i = 1, 2, \dots, N-1. \quad (1)$$

¹ This is due to the existence of a Lorentz-covariant energy-momentum tensor in Yang-Mills theory.

² For brevity vector gauge bosons are called “gluons” below. All Yang-Mills amplitudes are partial, associated to a single trace of (arbitrary) gauge group generators in the fundamental representation.

We will also introduce one additional pair of gluons, with the momenta p and q , and *opposite* helicities, $\mu = +1$ and $\nu = -1$, respectively. With all momenta assumed to be incoming into the scattering process, the momentum conservation is

$$\sum_{i=1}^{N-1} p_i + \sum_{i=1}^{N-1} q_i + p + q = \sum_{i=1}^{N-1} k_i + p + q = 0. \quad (2)$$

All momenta are on-shell (light-like). It is convenient to represent p and q as matrices which factorize into helicity spinor variables in the following way:

$$\not{p} = \sigma_p \tilde{\sigma}_p, \quad \not{q} = \sigma_q \tilde{\sigma}_q. \quad (3)$$

For real light-like momenta p and q , the limit $s_{pq} = (p + q)^2 = \langle pq \rangle [qp] \rightarrow 0$ at finite p and q constraints the respective three-momenta to a collinear configuration, *i.e.* both pointing in the same direction³. Here, however, we will be considering complex momenta, which allows two ways of reaching $s_{pq} = 0$:

$$[pq] \rightarrow 0 \text{ with } \langle pq \rangle \neq 0 : \quad \tilde{\sigma}_p \rightarrow x \tilde{\sigma}_q, \quad (4)$$

where x is an arbitrary number, and similarly,

$$\langle pq \rangle \rightarrow 0 \text{ with } [pq] \neq 0 : \quad \sigma_p \rightarrow x \sigma_q. \quad (5)$$

We will be using the following $2N$ -gluon Yang-Mills partial amplitudes

$$\begin{aligned} & A[p, N-1, 1, \pi(2, 3, \dots, N-2), 1, \rho(2, \dots, N-2), N-1, q] \\ & \equiv A[p, \mu = +1; p_{N-1}, \mu_{N-1}; \dots; p_{(N-2)\pi}, \mu_{(N-2)\pi}; q_1, \nu_1; \dots; q_{N-1}, \nu_{N-1}; q, \nu = -1], \end{aligned} \quad (6)$$

where $\pi, \rho \in S_{N-3}$ denote permutations of $N-3$ elements and $i_\pi \equiv \pi(i)$, $j_\rho \equiv \rho(j)$. We will also need the KLT kernel $S[\pi|\rho]$ introduced in [1,10,11]

$$S[\pi|\rho] \equiv S[\pi(2, \dots, N-2) | \rho(2, \dots, N-2)] = \prod_{i=2}^{N-2} \left(s_{1, i_\pi} + \sum_{j=2}^{i-1} \theta(i_\pi, j_\pi) s_{i_\pi, j_\pi} \right), \quad (7)$$

where $s_{i,j} \equiv (p_i + p_j)^2$ and $\theta(i_\pi, j_\pi) = 1$ if the ordering of the legs (i_π, j_π) and (i_ρ, j_ρ) is the same for $\pi(2, \dots, N-2)$ and $\rho(2, \dots, N-2)$, and zero otherwise⁴.

³ We are using standard notation of the helicity formalism [9].

⁴ Note, that the kernel (7) does not depend explicitly on the momenta p and q .

Theorem: The N -graviton amplitude in Einstein's gravity is given at the tree level by:

$$A_E[k_1, \lambda_1; \dots; k_{N-1}, \lambda_{N-1}; k_N = p + q, \lambda_N = +2] = \lim_{[pq] \rightarrow 0} \left(\frac{1}{2x} \right)^4 \frac{[pq]}{\langle pq \rangle} s_{pq}^2 \times \sum_{\pi, \rho \in S_{N-3}} S[\pi|\rho] A[p, N-1, 1, \pi(2, 3, \dots, N-2), 1, \rho(2, \dots, N-2), N-1, q] , \quad (8)$$

where the limit is defined in Eq. (4). Similarly,

$$A_E[k_1, \lambda_1; \dots; k_{N-1}, \lambda_{N-1}; k_N = p + q, \lambda_N = -2] = \lim_{\langle pq \rangle \rightarrow 0} (2x)^4 \frac{\langle pq \rangle}{[pq]} s_{pq}^2 \times \sum_{\pi, \rho \in S_{N-3}} S[\pi|\rho] A[p, N-1, 1, \pi(2, 3, \dots, N-2), 1, \rho(2, \dots, N-2), N-1, q] , \quad (9)$$

with the limit defined⁵ in Eq. (5).

Proof: In order to prove Eq. (8), we note that $[pq]s_{pq}^2 \sim s_{pq}^3$, therefore the limit $[pq] \rightarrow 0$ pushes the Yang Mills amplitude on the r.h.s. of (8) into a highly singular (triple “factorization pole”) kinematic configuration. In the first step, we factorize on the pole in the N -gluon channel shown in Fig.1, with the total momentum of:

$$l = \sum_{i=1}^{N-1} p_i + p = \frac{p-q}{2} , \quad l^2 = -\frac{s_{pq}}{4} . \quad (10)$$

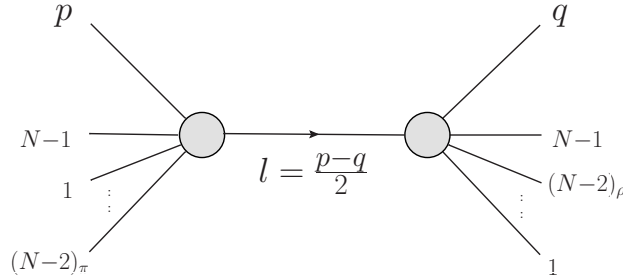


Figure 1: Factorization in the N -gluon channel.

Furthermore, the subamplitude on the left side of Fig.1 develops a pole in the two-gluon channel with

$$p_N = p - l = \frac{p+q}{2} = \frac{k_N}{2} , \quad p_N^2 = \frac{s_{pq}}{4} , \quad (11)$$

as shown in Fig.2. Similarly, the subamplitude on the right side of Fig.1 develops a pole in the two-gluon channel with

$$q_N = q + l = \frac{p+q}{2} = \frac{k_N}{2} , \quad q_N^2 = \frac{s_{pq}}{4} . \quad (12)$$

⁵ In the above relations, we omit constant factors involving Yang-Mills and gravitational coupling constants.

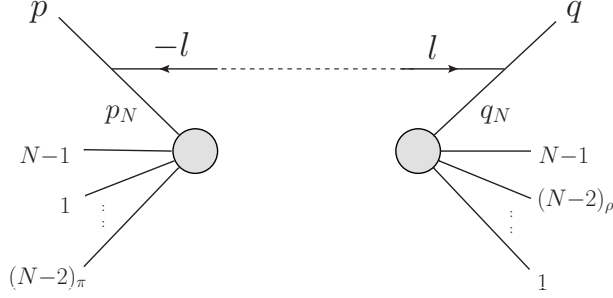


Figure 2: Factorization of subamplitudes.

It is easy to show that in the limit (4), there is a unique helicity configuration contributing to the triple pole

$$\begin{aligned}
A[p, N-1, 1, \pi(2, 3, \dots, N-2), 1, \rho(2, \dots, N-2), N-1, q] &\rightarrow \left(\frac{4}{s_{pq}}\right)^3 \times \\
&\times A[p^+, -l^-, -p_N^-] \times A[p_N, \mu_N = +1; N-1, 1, \pi(2, 3, \dots, N-2)] \\
&\times A[1, \rho(2, \dots, N-2), N-1; q_N, \nu_N = +1] \times A[q^-, l^+, -q_N^-] ,
\end{aligned} \tag{13}$$

where we used \pm superscripts to indicate the respective ± 1 vector boson helicities. In this limit, the three-gluon amplitudes reduce to:

$$A[p^+, -l^-, -p_N^-] = \frac{x^3}{2} \langle pq \rangle , \quad A[q^-, l^+, -q_N^-] = \frac{x}{2} \langle pq \rangle . \tag{14}$$

After inserting Eqs. (13) and (14) into Eq. (8), we obtain

$$\begin{aligned}
A_E[k_1, \lambda_1; \dots; k_{N-1}, \lambda_{N-1}; k_N, \lambda_N = +2] \\
= \sum_{\pi, \rho \in S_{N-3}} S[\pi|\rho] A[p_N, \mu_N = +1; N-1, 1, \pi(2, 3, \dots, N-2)] \\
\times A[1, \rho(2, \dots, N-2), N-1; q_N, \nu_N = +1] ,
\end{aligned} \tag{15}$$

which is the KLT formula for the N -graviton amplitude [1,10]. The proof of Eq. (9) proceeds in a similar way⁶.

Example: Three-graviton amplitude.

For $N = 3$, the KLT kernel is trivial and Eq. (8) reads

$$A_E[k_1, \lambda_1; k_2, \lambda_2; k_3 = p + q, \lambda_3 = +2] = \lim_{[pq] \rightarrow 0} \left(\frac{1}{2x}\right)^4 \frac{[pq]}{\langle pq \rangle} s_{pq}^2 A[p, 2, \{1, 1\}, 2, q] , \tag{16}$$

⁶ The amplitudes involving string dilatons can be obtained in a similar way, as collinear limits of Yang-Mills amplitudes involving two additional gauge bosons, but carrying identical helicities, *i.e.* $\mu = \nu = \pm 1$.

where we used curly brackets to indicate symmetrization in $\{p_1, q_1\}$ before setting $p_1 = q_1$. The symmetrization removes the collinear singularity at $p_1 = q_1$; it is necessary in the $N = 3$ case only. The r.h.s. of Eq.(16) can be rewritten by using the BCJ relation [12]

$$s_{pq}^2 A[p, 2, \{1, 1\}, 2, q] = s_{pq} (- s_{pq} A[p, 1, 2, 1, 2, q] + 2 s_{q1} A[p, 2, 1, 2, q, 1]) , \quad (17)$$

which is manifestly finite. The combination on the r.h.s. of the above equation appears in the zero string slope limit of the four-particle open-closed string disk amplitude involving two gravitons and two gauge bosons, *cf.* Eq. (3.40) of Ref. [13]:

$$s_{pq}^2 A[p, 2, \{1, 1\}, 2, q] = A[k_1, \lambda_1; k_2, \lambda_2; p, \mu; q, \nu] . \quad (18)$$

In this way, we find that the limit of Eq. (16) amounts to factorizing this Einstein-Yang-Mills amplitude in the s channel, on the graviton pole. In Ref. [7], we will show that the same conclusion holds for higher N : the triple pole limits of Yang-Mills amplitudes in Eqs. (8) and (9) correspond to the degeneration limit of disk amplitudes involving $N-1$ gravitons (closed strings) and two gauge bosons, *i.e.* open strings attached to the disk boundary. In this limit, the boundary shrinks to a point and the amplitude factorizes into N -graviton amplitude on the sphere times the amplitude for one of the gravitons to decay into a pair of gauge bosons. Now returning to the case of $N = 3$, we see from Eq. (17) that a non-vanishing amplitude requires two gravitons to carry opposite helicities, thus we set $\lambda_1 = +2$, $\lambda_2 = -2$. The amplitude (18) can be computed by substituting the well-known six-point NMHV Yang-Mills amplitudes [9] into the r.h.s. of Eq. (17), giving

$$A[k_1, \lambda_1 = +2; k_2, \lambda_2 = -2; p, \mu = +1; q, \nu = -1] = \frac{[1p][1q]}{\langle 1p \rangle \langle 1q \rangle} \frac{\langle 2q \rangle^4}{s_{pq}} . \quad (19)$$

Next, we substitute it to Eqs.(18) and (16), and take the limit by using

$$x = \frac{[1p]}{[1q]} = \frac{[2p]}{[2q]} , \quad k_3 = p + q = -k_1 - k_2 , \quad (20)$$

which yields the correct result:

$$A_E[k_1, \lambda_1 = +2; k_2, \lambda_2 = -2; k_3, \lambda_3 = +2] = \frac{[13]^6}{[12]^2 [23]^2} . \quad (21)$$

To summarize, we proposed a new representation of gravitational amplitudes at the tree level. It would be interesting to learn whether it can be extended beyond the tree level, to improve our understanding of loop corrections in quantum gravity.

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References

- [1] H. Kawai, D.C. Lewellen and S.H.H. Tye, “A Relation Between Tree Amplitudes Of Closed And Open Strings,” Nucl. Phys. B **269**, 1 (1986).
- [2] W. Siegel, “Hidden gravity in open string field theory,” Phys. Rev. D **49**, 4144 (1994). [hep-th/9312117].
- [3] F. Cachazo, S. He and E. Y. Yuan, “Scattering of Massless Particles in Arbitrary Dimensions,” Phys. Rev. Lett. **113**, 171601 (2014) [arXiv:1307.2199 [hep-th]].
- [4] F. Cachazo, S. He and E. Y. Yuan, “Scattering of Massless Particles: Scalars, Gluons and Gravitons,” JHEP **1407**, 033 (2014). [arXiv:1309.0885 [hep-th]].
- [5] F. Cachazo, S. He and E. Y. Yuan, “Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations,” [arXiv:1409.8256 [hep-th]].
- [6] S. Stieberger and T. R. Taylor, “Graviton as a Pair of Collinear Gauge Bosons,” Phys. Lett. B **739**, 457 (2014). [arXiv:1409.4771 [hep-th]].
- [7] S. Stieberger and T.R. Taylor, in preparation.
- [8] S. Weinberg and E. Witten, “Limits on Massless Particles,” Phys. Lett. B **96**, 59 (1980).
- [9] M.L. Mangano and S.J. Parke, “Multiparton amplitudes in gauge theories,” Phys. Rept. **200**, 301 (1991). [hep-th/0509223];
L.J. Dixon, “Calculating scattering amplitudes efficiently,” in Boulder 1995, QCD and beyond 539-582. [hep-ph/9601359].
- [10] Z. Bern, L.J. Dixon, M. Perelstein and J.S. Rozowsky, “Multileg one loop gravity amplitudes from gauge theory,” Nucl. Phys. B **546**, 423 (1999). [hep-th/9811140].
- [11] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, “The Momentum Kernel of Gauge and Gravity Theories,” JHEP **1101**, 001 (2011). [arXiv:1010.3933 [hep-th]].
- [12] Z. Bern, J.J.M. Carrasco and H. Johansson, “New Relations for Gauge-Theory Amplitudes,” Phys. Rev. D **78**, 085011 (2008). [arXiv:0805.3993 [hep-ph]].
- [13] S. Stieberger, “Open & Closed vs. Pure Open String Disk Amplitudes,” [arXiv:0907.2211 [hep-th]].